Exam 1 Solutions.

1. We use the equation $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$. Note that $f'(x) = 3x^2e^{x^3} + 3$. We also need to compute $f^{-1}(2)$. Since we can't easily solve $y = f(x)$ for x, we try to guess a value of x so that $f(x) = 2$; after a little trial-and-error we see that $f(0) = 2$. Therefore,
$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} = \frac{1}{3}.$$

2. We first take a logarithm of $y = f(x)$:
$$\ln y \quad = \quad \ln\left(\frac{(x^2+1)\sin^2 x}{\sqrt{x^3+2}}\right) = \ln(x^2+1) + 2\ln\sin x - \frac{1}{2}\ln(x^3+2).$$
Then we take the derivative of both sides: $\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{2\cos x}{\sin x} - \frac{3x^2}{2(x^3+2)}$. Then substituting $f(x)$ yields
$$f'(x) = y' = \left(\frac{(x^2+1)\sin^2 x}{\sqrt{x^3+2}}\right)\left(\frac{2x}{x^2+1} + \frac{2\cos x}{\sin x} - \frac{3x^2}{2(x^3+2)}\right).$$

3. We'll use the substitution $u = 1 + e^{3x}$, and so $du = 3e^{3x}\,dx$. Also, if $x = 0$ then $u = 1 + e^0 = 2$ and if $x = \ln 2$ then $u = 1 + e^{3\ln 2} = 1 + 8 = 9$. Therefore
$$\int_0^{\ln 2} \frac{e^{3x}}{1+e^{3x}}\,dx = \int_2^9 \frac{1}{3u}\,du = \frac{1}{3}\ln|u|\Big|_2^9 = \frac{1}{3}(\ln 9 - \ln 2).$$

4. We use logarithmic differentiation. Letting $y = (\sin x)^{(x^2+1)}$, we have $\ln y = (x^2+1)\ln(\sin x)$. Differentiating both sides (using the chain rule) yields
$$\frac{1}{y}\frac{dy}{dx} = 2x\ln(\sin x) + \frac{(x^2+1)\cos x}{\sin x}.$$
Multiply both sides by y to obtain $\frac{dy}{dx} = (\sin x)^{(x^2+1)}\left[2x\ln(\sin x) + \frac{(x^2+1)\cos x}{\sin x}\right]$.

5. Letting $P(t)$ be the population t years after Jan 01, 2000, the formula for exponential growth is $P(t) = P(0)e^{rt}$. Fill in the value $P(0) = 5000$, and solve for r using $P(5) = 5500$.
$$5500 = 5000e^{5r} \Rightarrow r = \frac{1}{5}\ln\left(\frac{5500}{5000}\right) = \frac{1}{5}\ln(1.1)$$
Then $P(t) = 5000e^{\ln(1.1)t/5} = 5000 \cdot 1.1^{t/5}$. Now set $t = 20$ to obtain the answer. $P(20) = 5000e^{4\ln(1.1)} = 5000(1.1)^4$.

6. Set $u = \ln x$. Then $du = \frac{1}{x}dx$, and the limits are $\ln(1) = 0$ and $\ln(e^{1/\sqrt{2}}) = 1/\sqrt{2}$. The integral is transformed to
$$\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-u^2}}du$$
which we recognize immediately to be $\arcsin u\big|_0^{1/\sqrt{2}} = \pi/4 - 0 = \pi/4$.

7. As $x \to \infty$, both $(\ln x)^2$ and $x \to \infty$, so this is indeterminate of form $\frac{\infty}{\infty}$. Thus we apply l'Hospital's Rule:
$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2\ln x \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x}$$
But both $2\ln x$ and $x \to \infty$ as $x \to \infty$, so this is still of indetereminate form $\frac{\infty}{\infty}$. So we apply l'Hospital's Rule again:
$$\lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = \lim_{x\to\infty} \frac{2}{x} = 0$$

8. We use integration by parts, with $u = x$ and $dv = \sin(2x)$. Then $du = dx$ and $v = \frac{-\cos(2x)}{2}$. Thus
$$\int x\sin(2x)dx = \frac{-x\cos(2x)}{2} - \int \frac{-\cos(2x)}{2}dx = \frac{-x\cos(2x)}{2} + \frac{\sin(2x)}{4} + C$$

9. Our goal is to do a u-substitution, with $u = \tan x$. Thus, $du = \sec^2(x)dx$ so we will leave one $\sec^2 x$ to become the du, and convert the other $\sec^2 x$ into $1 + \tan^2 x$. So we have:
$$\int \tan^{100} x \sec^4 x dx = \int \tan^{100} x\,(1 + \tan^2 x)\,\sec^2(x)dx = \int u^{100}(1+u^2)du$$
We integrate, and then back substitute to get
$$\int \tan^{100} x \sec^4 x dx = \int u^{100} + u^{102}du = \frac{u^{101}}{101} + \frac{u^{103}}{103} + C = \frac{\tan^{101} x}{101} + \frac{\tan^{103} x}{103} + C$$

10. Since $(x-1)$ is a polynomial of degree 1, then we get the term $A/(x-1)$. Secondly, $(x-5)$ is a polynomial of degree 1, with exponent 2, hence we get the terms $B/(x-5)$ and $C/(x-5)^2$. Finally, (x^2+1) is an irreducible polynomial of degree 2, so we get the term $(Dx+E)/(x^2+1)$. So, the correct form of the partial fraction decomposition of $f(x)$ is
$$\frac{A}{x-1} + \frac{B}{x-5} + \frac{C}{(x-5)^2} + \frac{Dx+E}{x^2+1}.$$

11. Let $L = \lim_{x\to 0^+} (\tan(x))^x$. Taking logarithm to both sides of the equation, we get
$$\ln(L) = \ln\left(\lim_{x\to 0^+} (\tan(x))^x\right) = \lim_{x\to 0^+} \ln\left((\tan(x))^x\right) = \lim_{x\to 0^+} x\ln(\tan(x)) = \lim_{x\to 0^+} \frac{\ln(\tan(x))}{1/x}$$
Since this is an indeterminate of the form $-\infty/\infty$, by L'Hopital's rule we get
$$\ln(L) = \lim_{x\to 0^+} \frac{\sec^2(x)/(\tan(x)}{-1/x^2} = \lim_{x\to 0^+} \frac{-x^2}{\sin(x)\cos(x)}$$
This is again an indeterminate of the form $0/0$, by L'Hopital's rule we get
$$\ln(L) = \lim_{x\to 0^+} \frac{-2x}{\cos^2(x) - \sin^2(x)} = 0.$$
Since $\ln(L) = 0$ we get $L = 1$.

12. Lets calculate the integral using the partial fraction method. The right form for the partial fraction decomposition of $10/(x-3)(x^2+1)$ is

$$\frac{10}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1}.$$

Hence, $10 = A(x^2+1) + (Bx+C)(x-3)$. Setting $x = 3$, we get $10 = 10A$, this is, $A = 1$. Setting $x = 0$, we get $10 = 1(1) + C(-3)$, this is, $C = -3$. Setting $x = 1$, we get $10 = (1)(2) + (B-3)(-2)$, this is, $B = -1$. So

$$\int \frac{10}{(x-3)(x^2+1)} dx = \int \frac{1}{(x-3)} + \frac{-x-3}{(x^2+1)} dx = \int \frac{1}{(x-3)} dx + \int \frac{-x}{(x^2+1)} dx + \int \frac{-3}{(x^2+1)} dx$$

Now, making the substitution $w = x - 3$ on the first integral, we get

$$\int \frac{1}{(x-3)} = \ln(|x-3|) + C_1.$$

For the second integral, let $u = x^2 + 1$, $du = 2xdx$. Hence

$$\int \frac{-x}{(x^2+1)} dx = -\frac{1}{2} \int \frac{1}{u}\, du = \ln|u| = -\frac{1}{2} \ln(|x^2+1|) + C_2.$$

Finally, for the third integral, we get

$$\int \frac{-3}{(x^2+1)} dx = -3 \int \frac{1}{(x^2+1)} dx = -3\arctan(x) + C_3$$

This shows,

$$\int \frac{10}{(x-3)(x^2+1)} dx = \ln(|x-3|) - \frac{1}{2}\ln(|x^2+1|) - 3\arctan(x) + C.$$

13. Since the argument of the integral is of the form $\sqrt{x^2+a^2}$, we make the substitution $x = 3\tan(\theta) \Rightarrow dx = 3\sec^2(\theta)d\theta$ (and remember to change the limits of integration), which yields,

$$\int_0^3 \frac{1}{\sqrt{x^2+9}} dx = \int_0^{\pi/4} \frac{3\sec^2(\theta)}{\sqrt{9\tan^2(\theta)+9}} d\theta$$

We factor the 9 out of the square root and then make the substitution $1 + \tan^2 = \sec^2$ to get

$$\int_0^3 \frac{1}{\sqrt{x^2+9}} dx = \int_0^{\pi/4} \frac{3\sec^2(\theta)}{3\sqrt{\sec^2(\theta)}} d\theta = \int_0^{\pi/4} \sec(\theta)d\theta = \ln|\sec(\theta) + \tan(\theta)||_0^{\pi/4},$$

where in the last equality, we applied the last formula from the 'list of useful trigonometric formulas' at the end of the exam. Evaluating, we get

$$\int_0^3 \frac{1}{\sqrt{x^2+9}} dx = \ln|2/\sqrt{2}+1| - \ln|1+0| = \ln|\sqrt{2}+1|$$